MATRICES

DEFINITION

A rectangular arrangement of elements in rows and columns, is called a matrix. Such a rectangular arrangement of numbers is enclosed by small ( ) or big [ ] brackets. Generally a matrix is represented by a capital latter A, B, C......... etc. and its element are represented by small letters a, b, c, x, y etc.

Following are some examples of a matrix :

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{bmatrix} 1 & 5 & 3 \\ 4 & 0 & 2 \end{bmatrix} \quad C = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad D = [1, 5, 6], \quad E = [5]$$

ORDER OF MATRIX

A matrix which has m rows and n columns is called a matrix of order m × n, and its represented by

$A_{m\times n}$ or $A = [a_{ij}]_{m\times n}$

It is obvious to note that a matrix of order m × n contains mn elements. Every row of such a matrix contains n elements and every column contains m elements.

TYPES OF MATRICES

Row matrix

If in a matrix, there is only one row, then it is called a Row Matrix.

Thus $A = [a_{ij}]_{m\times n}$ is a row matrix if m = 1

Column Matrix

If in a matrix, there is only one column, then it is called a column matrix.

Thus $A = [a_{ij}]_{m\times n}$ is a column matrix if n = 1.

Square matrix

If number of rows and number of columns in a matrix are equal, then it is called a square matrix.

Thus $A = [a_{ij}]_{m\times n}$ is a square matrix if m = n.

Note : (a) If $m \neq n$ then matrix is called a rectangular matrix.

(b) The elements of a square matrix A for which i = j i.e., $a_{11}, a_{22}, a_{33}, ...a_{nn}$ are called principal diagonal elements and the line joining these elements is called the principal diagonal or leading diagonal of matrix A.

(c) Trace of a matrix : The sum of principal diagonal elements of a square matrix A is called the trace of matrix A which is denoted by trace A. Trace A = $a_{11} + a_{22} + ...a_{nn}$

Singleton matrix

If in a matrix there is only one element then it is called singleton matrix.

Thus $A = [a_{ij}]_{m\times n}$ is a singleton matrix if m = n = 1.

Null or zero matrix

If in a matrix all the elements are zero then it is called a zero matrix and it is generally denoted by O.

Thus $A = [a_{ij}]_{m\times n}$ is a zero matrix if $a_{ij} = 0$ for all i and j.

Diagonal matrix

If all elements except the principal diagonal in a square matrix are zero, it is called a diagonal matrix.

Thus a square matrix $A = [a_{ij}]$ is a diagonal matrix if $a_{ij} = 0$, when $i \neq j$.

Note : (a) No element of principal diagonal in diagonal matrix is zero.

(b) Number of zero is a diagonal matrix is given by $n^2 - n$ where n is a order of the matrix.

Scalar Matrix

If all the elements of the diagonal in a diagonal matrix are equal, it is called a scalar matrix.

Thus a square matrix A $[a_{ij}]$ is a scalar matrix is

$$a_{ij} = \begin{cases} 0 & i \neq j \\ k & i = j \end{cases}$$ where k is a constant.

Unit matrix

If all elements of principal diagonal in a diagonal matrix are 1, then it is called unit matrix. A unit matrix of order n is denoted by I_n.

Thus a square matrix

$A = [a_{ij}]$ is a unit matrix if $a_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

Note : Every unit matrix is a scalar matrix.

Triangular matrix

A square matrix $[a_{ij}]$ is said to be triangular if each element above or below the principal diagonal is zero. It is of two types -

(a) Upper triangular matrix : A square matrix $[a_{ij}]$ is called the upper triangular matrix, if $a_{ij} = 0$ when $i > j$.

(b) Lower triangular matrix : A square matrix $[a_{ij}]$ is called the lower triangular matrix, if $a_{ij} = 0$ when $i < j$

Note : Minimum number of zero in a triangular matrix is given by $\frac{n(n-1)}{2}$ where n is order of matrix.

Equal matrix

Two matrices A and B are said to be equal if they are of same order and their corresponding elements are equal.

Singular matrix

Matrix A is said to be singular matrix if its determinant $|A| = 0$, otherwise non-singular matrix i.e.,

If det $|A| = 0 \Rightarrow$ singular and det $|A| \neq 0 \Rightarrow$ non-singular

ADDITION AND SUBTRACTION OF MATRICES

If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are two matrices of the same order then their sum A + B is a matrix whose each element is the sum of corresponding elements.

i.e., $A + B = [a_{ij} + b_{ij}]_{m \times n}$

$A - B$ is defined as $A - B = [a_{ij} - b_{ij}]_{m \times n}$

Note : Matrix addition and subtraction can be possible only when matrices are of same order.

Properties of matrices addition

If A, B and C are matrices of same order, then-

(i) A + B = B + A (Commutative Law)

(ii) (A + B) + C = A + (B + C) (Associative law)

(iii) A + O = O + A = A, where O is zero matrix which is additive identity of the matrix.

(iv) A + (–A) = 0 = (–A) + A where (–A) is obtained by changing the sign of every element of A which is additive inverse of the matrix

(v) $\left.\begin{array}{l} A+B=A+C \\ B+A=C+A \end{array}\right\} \Rightarrow B = C$ (Cancellation law)

(vi) Trace $(A \pm B)$ = trace $(A) \pm$ trace (B)

SCALAR MULTIPLICATION OF MATRICES

Let $A = [a_{ij}]_{m\times n}$ be a matrix and k be a number then the matrix which is obtained by multiplying every element of A by k is called scalar multiplication of A by k and it denoted by kA.

Thus $A = [a_{ij}]_{m\times n} \Rightarrow kA = [ka_{ij}]_{m\times n}$

Properties of scalar multiplication

If A, B are matrices of the same order and m, n are any numbers, then the following results can be easily established.

(i) $m(A + B) = mA + mB$ (ii) $(m + n)A = mA + nA$ (iii) $m(nA) = (mn)A = n(mA)$

MULTIPLICATION OF MATRICES

If A and B be any two matrices, then their product AB will be defined only when number of column in A is equal to the number of rows in B. If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$ then their product $AB = C = [c_{ij}]$, will be matrix of order $m \times p$, where

$$(AB)_{ij} = C_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}$$

Properties of matrix multiplication

If A, B and C are three matrices such that their product is defined, then

(i) $AB \neq BA$ (Generally not commutative)

(ii) $(AB)C = A(BC)$ (Associative Law)

(iii) $IA = A = AI$ I is identity matrix for matrix multiplication

(iv) $A(B + C) = AB + AC$ (Distributive law)

(v) If $AB = AC \not\Rightarrow B = C$ (cancellation Law is not applicable)

(vi) If $AB = 0$ It does not mean that $A = 0$ or $B = 0$, again product of two non-zero matrix may be zero matrix.

(vii) trance (AB) = trance (BA)

Note : (i) The multiplication of two diagonal matrices is again a diagonal matrix.

(ii) The multiplication of two triangular matrices is again a triangular matrix.

(iii) The multiplication of two scalar matrices is also a scalar matrix.

(iv) If A and B are two matrices of the same order, then

(a) $(A + B)^2 = A^2 + B^2 + AB + BA$

(b) $(A - B)^2 = A^2 + B^2 - AB - BA$

(c) $(A - B)(A + B) = A^2 - B^2 + AB - BA$

(d) $(A + B)(A - B) = A^2 - B^2 - AB + BA$

(e) $A(-B) = (-A)B = -(AB)$

Positive Integral powers of a matrix

The positive integral powers of a matrix A are defined only when A is a square matrix.

Also then $A^2 = A.A \quad A^3 = A.A.A = A^2A$

Also for any positive integers m, n

(i) $A^m A^n = A^{m+n}$ (ii) $(A^m)^n = A^{mn} = (A^n)^m$

(iii) $I^n = I, I^m = I$ (iv) $A^\circ = I_n$ where A is a square matrices of order n.

TRANSPOSE OF MATRIX

If we interchange the rows to columns and columns to rows of a matrix A, then the matrix so obtained is called the transpose of A and it is denoted by

A^T or A^t or A'

From this definition it is obvious to note that

(i) Order of A is $m \times n \Rightarrow$ order of A^T is $n \times m$

(ii) $(A^T)_{ij} = (A)_{ji}$, " i, j)

Properties of Transpose

If A, B are matrices of suitable order then

(i) $(A^T)^T = A$

(ii) $(A + B)^T = A^T + B^T$

(iii) $(A - B)^T = A^T - B^T$

(iv) $(kA)^T = kA^T$

(v) $(AB)^T = B^T A^T$

(vi) $(A_1 A_2 \ldots A_n)^T = A_n^T \ldots A_2^T A_1^T$

(vii) $(A^n)^T = (A^T)^n$, $n \in N$

SYMMETRIC AND SKEW-SYMMETRIC MATRIX

(a) Symmetric matrix : A square matrix $A = [a_{ij}]$ is called symmetric matrix if $a_{ij} = a_{ji}$ for all $\hat{i} = \hat{j}$ or $A^T = A$.

Note : (i) Every unit matrix and square zero matrix are symmetric matrices.

(ii) Maximum number of different element in a symmetric matrix is $\frac{n(n+1)}{2}$

(b) Skew-symmetric matrix : A square matrix $A = [a_{ij}]$ is called skew-symmetric matrix if

$a_{ij} = -a_{ji}$ for all i, j or $A^T = -A$

Note : (i) All principal diagonal elements of a skew-symmetric matrix are always zero because for any diagonal element - $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$

(ii) Trace of a skew symmetric matrix is always 0

Properties of symmetric and skew-symmetric matrices

(i) If A is a square matrix, then $A + A^T$, AA^T, A^TA are symmetric matrices while $A - A^T$ is skew-symmetric matrices.

(ii) If A, B are two symmetric matrices, then-

(a) $A \pm B$, $AB + BA$ are also symmetric matrices.

(b) $AB - BA$ is a skew-symmetric matrix.

(c) AB is a symmetric matrix when $AB = BA$

(iii) If A, B are two skew-symmetric matrices, then-

(a) $A \pm B$, $AB - BA$ are skew-symmetric matrices.

(b) $AB + BA$ is a symmetric matrix.

(iv) If A is a skew-symmetric matrix and C is a column matrix, then $C^T AC$ is a zero matrix.

(v) Every square matrix A can be uniquely be expressed as sum of a symmetric and skew symmetric matrix i.e.,

$$A = \left[\frac{1}{2}(A + A^T)\right] + \left[\frac{1}{2}(A - A^T)\right]$$

DETERMINANT OF A MATRIX

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix, then its determinant, denoted by |A| or det. (A) is

defined as $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Properties of the determinant of a matrix

(i) |A| exist $\Leftrightarrow$ A is a square matrix

(ii) $|AB| = |A|\ |B|$

(iii) $|A^T| = |A|$

(iv) $|kA| = k^n |A|$, if A is a square matrix of order n.

(v) If A and B are square matrices of same order then |AB| = |BA|

(vi) If A is skew symmetric matrix of odd order then |A| = 0

(vii) If $A = \text{diag}(a_1, a_2, ...a_n)$ then $|A| = a_1 a_2 ....a_n$

(viii) $|A|^n = |A^n|$, $n \in N$

ADJOINT OF A MATRIX

If every element of a square matrix A be replaced by its cofactor in |A|, then the transpose of the matrix so obtained is called the adjoint of A and it is denoted by adj A

Thus if $A = [a_{ij}]$ be a square matrix and C_{ij} be the cofactor of a_{ij} in |A|, then

$\text{adj } A = [C_{ij}]^T$

$\Rightarrow (\text{adj } A)_{ij} = C_{ij}$

Hence if $A = \begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ .... & .... & .... & .... \\ a_{n1} & a_{n2} & .... & a_{nn} \end{bmatrix}$, then $\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & .... & C_{1n} \\ C_{21} & C_{22} & .... & C_{2n} \\ .... & .... & .... & .... \\ C_{n1} & C_{n2} & .... & C_{nn} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & .... & C_{n1} \\ C_{12} & C_{22} & .... & C_{n2} \\ .... & .... & .... & .... \\ C_{1n} & C_{2n} & .... & C_{nn} \end{bmatrix}$

Properties of Adjoint Matrix

If A, B are square matrices of order n and I_n is corresponding unit matrix, then

(i) $A (\text{adj } A) = |A|\ I_n = (\text{adj } A)\ A$

(Thus A (adj A) is always a scalar matrix)

(ii) $|\text{adj } A| = |A|^{n-1}$

(iii) $\text{adj}(\text{adj } A) = |A|^{n-2} A$

(iv) $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$

(v) $\text{adj}(A^T) = (\text{adj } A)^T$

(vi) $\text{adj}(AB) = (\text{adj } B)(\text{adj } A)$

(vii) $\text{adj}(A^m) = (\text{adj } A)^m, m \in N$

(viii) $\text{adj}(kA) = k^{n-1}(\text{adj } A), k \in R$

(ix) $\text{adj}(I_n) = I_n$

(x) adj 0 = 0

(xi) A is symmetric $\Rightarrow$ adj A is also symmetric.

(xii) A is diagonal $\Rightarrow$ adj A is also diagonal.

(xiii) A is triangular $\Rightarrow$ adj A is also triangular.

(xiv) A is singular $\Rightarrow$ |adj A| = 0

INVERSE MATRIX

If A and B are two matrices such that

$$AB = I = BA$$

then B is called the inverse of A and it is denoted by A^{-1}. Thus

$$A^{-1} = B \Leftrightarrow AB = I = BA$$

Further we may note from above property (i) of adjoint matrix that if $|A| \neq 0$, then

$$A\frac{\text{adj}(A)}{|A|} = I = \frac{(\text{adj } A)}{|A|}A \qquad \Rightarrow \qquad A^{-1} = \frac{1}{|A|}\text{adj } A$$

Thus A^{-1} exists $\Leftrightarrow$ $|A| \neq 0$.

Note :

(i) Matrix A is called invertible if A^{-1} exists.

(ii) Inverse of a matrix is unique.

Properties of Inverse Matrix

(i) $(A^{-1})^{-1} = A$

(ii) $(A^T)^{-1} = (A^{-1})^T$

(iii) $(AB)^{-1} = B^{-1}A^{-1}$

(iv) $(A^n)^{-1} = (A^{-1})^n, n \in N$

(v) $\text{adj}(A^{-1}) = (\text{adj } A)^{-1}$

(vi) $|A^{-1}| = \dfrac{1}{|A|} = |A|^{-1}$

(vii) $A = \text{diag}(a_1, a_2, \ldots, a_n) \Rightarrow A^{-1} = \text{diag}(a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1})$

(viii) A is symmetric $\Rightarrow$ A^{-1} is also symmetric.

(ix) A is diagonal $|A| \neq 0$ $\Rightarrow$ A^{-1} is also diagonal.

(x) A is scalar matrix $\Rightarrow$ A^{-1} is also scalar matrix.

(xi) A is triangular $|A| \neq 0$ $\Rightarrow$ A^{-1} is also triangular.

SOME IMPORTANT CASES OF MATRICES

Orthogonal Matrix

A square matrix A is called orthogonal if

$AA^T = I = A^TA$; i.e., if $A^{-1} = A^T$

Idempotent matrix

A square matrix A is called an idempotent matrix if $A^2 = A$

Involutory Matrix

A square matrix A is called an involutory matrix if $A^2 = I$ or $A^{-1} = A$

Nilpotent matrix

A square matrix A is called a nilpotent matrix if there exist a $p \in N$ such that $A^P = 0$

Hermition matrix

A square matrix A is skew-Hermition matrix if $A^q = A$; i.e., $a_{ij} = -\bar{a}_{ji}$ " i, j

Skew hermitian matrix

A square matrix A is skew-hermition is $A = -A^q$ i.e., $aij = -\bar{a}_{ji}$ " i, j

Period of a matrix

If for any matrix A $A^{k+1} = A$

then k is called period of matrix (where k is a least positive integer)

Differentiation of matrix

If $A = \begin{bmatrix} f(x) & g(x) \\ h(x) & \ell(x) \end{bmatrix}$

then $\frac{dA}{dx} = \begin{bmatrix} f'(x) & g'(x) \\ h'(x) & \ell'(x) \end{bmatrix}$ is a differentiation of matrix A

Submatrix

Let A be m × n matrix, then a matrix obtained by leaving some rows or columns or both of a is called a sub matrix of A

Rank of a matrix

A number r is said to be the rank of a m × n matrix A if

(a) every square sub matrix of order (r + 1) or more is singular and

(b) there exists at least one square submatrix of order r which is non-singular.

Thus, the rank of matrix is the order of the highest order non-singular sub matrix.

We have |A| = 0 therefore r (A) is less then 3, we observe that $\begin{bmatrix} 5 & 6 \\ 4 & 5 \end{bmatrix}$ is a non-singular square sub matrix of order 2 hence r (A) 2.

Note :

(i) The rank of the null matrix is zero.

(ii) The rank of matrix is same as the rank of its transpose i.e., $r(A) = r(A^T)$

(iii) Elementary transformation of not alter the rank of matrix.